LECTURE 18 DERIVATIVES OF EXPONENTIAL FUNCTIONS

Derivatives of a^u and $\log_a u$

About a^x. Now, suppose $f(x) = a^x$ where $a > 0$. We want to calculate $f'(x)$.

Clearly, by the exponentiation trick,

$$f(x) = a^x = e^{\ln a^x} = e^{x \ln a}.$$

Remember here that a is a constant, and so is $\ln(a)$. To find $f'(x)$, we do chain rule.

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \frac{d}{dx}(x \ln a) = a^x \ln a.$$

We can verify that

$$\frac{d}{dx} e^x = e^x \ln e = e^x.$$

How about $f(x) = a^{u(x)}$ where $a > 0$ and u is a differentiable function of x? By chain rule, the inner function is $u(x)$ and outer function is a^x.

$$\frac{d}{dx} f(x) = \frac{d}{dx} a^{u(x)} = \left(a^{u(x)} \ln a\right) \frac{du}{dx}.$$

Always remember, $\ln a$ is a constant, not a function of anything.

About $\log_a x$. Then, consider $g(x) = \log_a x$, $a > 0$. We again want $g'(x)$. We utilize the change of base formula,

$$g(x) = \log_a x = \frac{\ln x}{\ln a},$$

where again $\ln a$ is a constant. Thus,

$$g'(x) = \frac{1}{\ln a} \frac{d}{dx}(\ln x) = \frac{1}{x \ln a}.$$

Furthermore, if u is a differentiable function of x, then

$$\frac{d}{dx} \log_a u(x) = \frac{1}{u(x) \ln a} \frac{du}{dx}.$$

Another justification for the power rule. Consider the function $f(x) = x^n$. Then, by the exponentiation trick (and bringing out the power), we can write

$$f(x) = x^n = e^{n \ln x}.$$

Then, we can find the derivative via the differentiation rules found above. By chain rule,

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{n \ln x} = e^{n \ln x} \frac{d}{dx}(n \ln x) = x^n \left(\frac{n}{x}\right) = nx^{n-1}.$$

Example. Harder problem. Find the derivative of $f(x) = x^x$.

There are two ways to do this, all of which you should know by now.

Case 1. We do the exponentiation trick.

$$f(x) = x^x = e^{\ln x^x} = e^{x \ln x}.$$

Then, we differentiate using chain rule.

$$\frac{d}{dx} f(x) = \frac{d}{dx} e^{x \ln x} = e^{x \ln x} \frac{d}{dx}(x \ln x) = x^x \left(\ln x + x \cdot \frac{1}{x}\right) = x^x (\ln x + 1).$$

Case 2. We do implicit differentiation. First, take ln of both sides.

$$\ln\left(f\left(x\right)\right) = x\ln x.$$

Then, we perform $\frac{d}{dx}$ on both sides.

$$\frac{d}{dx}\ln\left(f\left(x\right)\right) = \frac{d}{dx}\left(x\ln x\right)$$

You realise that the LHS has inner function $f\left(x\right)$ and outer function $\ln\left(x\right)$. Thus, the above implies

$$\begin{aligned}\frac{1}{f\left(x\right)}\frac{d}{dx}f\left(x\right) &= \left(\ln x + x\cdot\frac{1}{x}\right)\\ \implies \frac{d}{dx}f\left(x\right) &= f\left(x\right)\left(\ln x+1\right)\\ &= x^{x}\left(\ln x+1\right).\end{aligned}$$

Same answer as the first approach.

Theorem.

$$e = \lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}.$$

Proof. The proof may be a little top-to-bottom, as in, pretty tricky. We don't start with the limit as stated. Instead, we try to show

$$\ln\left(\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}\right) = 1.$$

Noting that $\ln\left(x\right)$ is a continuous function, we can definitely write the LHS as (by definition of continuity, $\ln\left(\lim_{x\to a}x\right) = \lim_{x\to a}\ln\left(x\right)$)

$$\begin{aligned}\ln\left(\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}\right) &= \lim_{x\to 0}\ln\left(1+x\right)^{\frac{1}{x}}\\ &= \lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}\end{aligned}$$

This starts to look like a derivative, by adding a new term $\ln\left(1\right) = 0$,

$$\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x} = \lim_{x\to 0}\frac{\ln\left(1+x\right)-\ln\left(1\right)}{x} = \left(\frac{d}{dx}\ln\left(x\right)\right)|_{x=1} = \frac{1}{x}\,|_{x=1} = 1$$

and we are done. □

Example. Evaluate the limit $\lim_{x\to 0}\left(1+2x\right)^{\frac{1}{x}}$.

Proof. Certainly, we want to make use of the theorem. How? The idea is to make sure the $2x$ here matches with the exponent as $\frac{1}{2x}$, by doing some tricks. Note that

$$\left(1+2x\right)^{\frac{1}{x}} = \left(\left(1+2x\right)^{\frac{1}{2x}}\right)^{2}.$$

Thus,

$$\begin{aligned}\lim_{x\to 0}\left(1+2x\right)^{\frac{1}{x}} &= \lim_{x\to 0}\left(\left(1+2x\right)^{\frac{1}{2x}}\right)^{2}\\ &= \left(\lim_{x\to 0}\left(1+2x\right)^{\frac{1}{2x}}\right)^{2}\\ &\overset{y=2x}{=} \left(\lim_{y\to 0}\left(1+y\right)^{\frac{1}{y}}\right)^{2}\\ &= e^{2}\end{aligned}$$

□

Example. Show that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$.

Solution. One can replace n by y. Then,

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^y \\
&= \lim_{z\to 0}(1+xz)^{\frac{1}{z}} \\
&= \lim_{z\to 0}\left((1+xz)^{\frac{1}{xz}}\right)^x \\
&= \left(\lim_{z\to 0}(1+xz)^{\frac{1}{xz}}\right)^x \\
&\overset{x\neq 0}{=} \left(\lim_{w\to 0}(1+w)^{\frac{1}{w}}\right)^x \\
&= e^x.
\end{aligned}$$